

# On the dual variables description of Yang-Mills theory

A.G. Shuvaev

Petersburg Nuclear Physics Institute, Gatchina, S.Petersburg, 188300,  
Russia

## Abstract

The partition function of four dimensional  $SO(4)$  Yang-Mills theory is rewritten in terms of variables admitting straightforward relation to the partition function of pure 4D gravity. The gauge action turns into first-order Hilbert-Palatini action for Einstein gravity with a simple extra term added. The proposed relation can be substantiated as a duality for the partition functions provided a special gauge is imposed for the gravity. The same method allows to find a closed expression for the partition function of the  $SO(4)$  gauge theory.

## 1 Introduction

Among modern approaches developing relationship between gravity and Yang-Mills theory there are those based on the first order formulation of general relativity [1, 2, 3]. They reformulate the gravity in terms of spin connection rather than spacetime metric [4] and provides a framework to treat it as diffeomorphism invariant gauge theory, identifying the spin connection with  $SU(2)$  gauge field. On the other hand the first order formalism enables to rewrite Yang-Mills theory through gauge-invariant dual variables [5, 6, 7, 8], which is of interest from the viewpoint of unified description of all interactions. The gravity action is quite different compared to those of Yang-Mills theories that are believed to be responsible for strong, weak and electromagnetic forces. Therefore if there exists a change of variables that brings the action of the gauge theory to the form close to the Einstein-Hilbert one or vice versa it could be a way to unify both the theories. Besides, such a transformation is interesting in itself as getting further insight into the nature of Yang-Mills theory.

The variable  $B_{\mu\nu}$  dual to the gauge field  $A_\mu$  is introduced as an auxiliary variable in the action [2, 8]

$$S_B = \int d^4x \left( -\frac{g^2}{2} \text{Tr} B_{\mu\nu} B_{\mu\nu} + \frac{i}{2} \varepsilon^{\mu\nu\lambda\sigma} \text{Tr} B_{\mu\nu} G_{\lambda\sigma} \right), \quad (1)$$

which, being varied with respect  $B_{\mu\nu}$ , returns the Yang-Mills action. Here  $G_{\mu\nu}$  is the standard Yang-Mills field strength and  $\varepsilon^{\mu\nu\lambda\sigma}$  is antisymmetric tensor. The basic feature of this form is that it makes the integral over the field  $A_\mu$  to be a Gaussian. It can be done explicitly with the result given by the action

$$S_{BF} = \frac{i}{2} \int d^4x \varepsilon^{\mu\nu\lambda\sigma} \text{Tr } B_{\mu\nu} G_{\lambda\sigma}(\bar{A}) \quad (2)$$

evaluated at the solution  $\bar{A}_\mu$  of the equation of motion,  $\varepsilon^{\mu\nu\lambda\sigma} [D_\nu(\bar{A}), B_{\lambda\sigma}] = 0$ , where  $D_\nu(A)$  is a covariant derivative. The action  $S_{BF}$  turns out to be a functional of the variable  $B_{\mu\nu}$ , which can be shown to be related to the metric of the dual color space  $g_{\mu\nu}$  and some special extra metric  $h_{ij}$  associated to  $SU(2)$  gauge group. Taken alone  $S_{BF}$  defines topological "BF theory" [9]. If, in addition, the metric  $h_{ij}$  is assumed to be trivial,  $h_{ij} = \delta_{ij}$ , the action (2) reduces to the standard Einstein-Hilbert one  $R\sqrt{g}$ . The first term in the total action breaks the general invariance and plays the role of 'aether' [8].

The starting point of this paper is to chose as dual variables the set of fourvectors  $e_\mu^A$  including  $4 \times 4 = 16$  independent components. Upon putting  $B_{\mu\nu}^{AB} = e_\mu^A e_\nu^B - e_\nu^A e_\mu^B$  the second term in the formula (1) turns into Hilbert-Palatini action with  $e_\mu^A$  playing role of tetrad. At first glance, dealing with  $e_\mu^A$  we loss at once the gauge invariance of the dual variables as well as the Gauss integral over them. Indeed, they are the vectors under the gauge transformation,  $e_\mu^A(x) \rightarrow R^{AB}(x) e_\mu^B(x)$ , where  $R \in SO(4)$  in Euclidean case (it would be a local Lorentz transformation in Minkowski space). Substituting tetrad in the term  $B_{\mu\nu}^2$  makes the integral over  $e_\mu^A$  to be non-Gaussian. The important fact however is that there are no derivatives of the  $e_\mu^A$  fields in it, so it is just the product of usual finite dimensional integrals at each point  $x$ . It is the main property the paper is based on. It enables to calculate the integral and to relate the result with the partition function of the gauge field. Taking the same integrals in the opposite order and starting at first with the integral over field  $A_\mu$ , which is still Gaussian, we arrive at the expression, which turns into gravity action, when the redundant gauge degrees of freedom inherited in the vectors  $e_\mu^A$  are integrated out. These topics are discussed in the Sections 2 and 3. The Section 4 is devoted to the modification of these results due to various forms of quantum measure adopted for gravity functional integral. It is argued in the Section 5 that the additional term appearing in the gravity action and spoiling its general coordinate invariance can be naturally interpreted as gauge fixing. It allows to establish gauge/gravity connection, or duality, at least for the partition functions.

It is shown in the Section 6 that the partition function of  $SO(4)$  gauge field can be rewritten through the dual variables in a way, where it turns completely into the product of independent integrals at each point  $x$ . Thus the gluon partition function looks like that calculated for ensemble of uncorrelated objects.

Note lastly that  $SO(4)$  partition function we have dealt with throughout the paper is simply related to  $SU(2)$  one,  $Z_{SO(4)} = Z_{SU(2)}^2$ .

## 2 Integral over tetrad

Here we outline the main idea leaving the details as well as some important modifications for the next section. We take the  $SO(4)$  gauge group with the gauge field  $A_\mu = A_\mu^{AB} T^{AB}$ , where  $\mu = 1, \dots, 4$  and  $T^{AB}$ ,  $A, B = 1, \dots, 4$  are the generators of 4D rotations,  $T^{AB} = -T^{BA}$ . The field strength tensor reads

$$G_{\mu\nu}^{AB}(A) = \left( \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \right)^{AB}.$$

Introducing auxiliary fields  $e_\mu^A(x)$  playing further the role of tetrad and tensor

$$\Sigma_{\mu\nu}^{AB} = e_\mu^A e_\nu^B - e_\nu^A e_\mu^B$$

we present the partition function for the  $SO(4)$  gauge field through the following functional integral

$$Z = \int DA_\mu D e_\mu^A \exp \int d^4x \left[ -\mu^4 (e_\mu^A e_\mu^A)^2 + iM^2 \tilde{G}(A) \cdot \tilde{\Sigma} \right], \quad (3)$$

where

$$\tilde{G}(A) \cdot \tilde{\Sigma} \equiv \varepsilon^{\mu\nu\lambda\sigma} \varepsilon^{ABCD} G_{\mu\nu}^{AB}(A) \Sigma_{\lambda\sigma}^{CD},$$

and  $\mu$  and  $M$  are two arbitrary mass parameters. To prove that the expression (3) is really coincides with the gluon partition function we directly calculate the functional integral over tetrad. For this purpose we shall treat it as a limit of multiple integral over discretized space,

$$\begin{aligned} Z[A] = & \prod_x \int d e_\mu^A(x) \exp \sum_x \left[ -\mu^4 (e_\mu^A(x) e_\mu^A(x))^2 \Delta x^4 \right. \\ & \left. + iM^2 \tilde{G}(x) \cdot \tilde{\Sigma}(x) \Delta x^4 \right] \end{aligned} \quad (4)$$

( $G(x) = G(A(x))$ ,  $\Sigma(x) = \Sigma(e_\mu^A(x))$ ). When the separation  $\Delta x \rightarrow 0$ , the multiplicity, that is the number of finite-dimensional integrals located at points  $x$ , goes to infinity while the Riemann sum turns into continuous integral for the action in the exponent.

The crucial property the subsequent analysis is based on is the absence of derivatives of the auxiliary fields  $e_\mu^A(x)$  in the action. It makes the integrations over  $e_\mu^A(x)$  to be independent from each other. In this context the integral (4) can be naturally thought of as an averaging of the action functional over ensemble of uncorrelated random variables,

$$Z[A] = \langle \langle \exp \sum_x iM^2 \tilde{G}(x) \cdot \tilde{\Sigma}(x) \Delta x^4 \rangle \rangle,$$

with the function

$$P(e_\mu^A) = \exp[-\mu^4 (e_\mu^A e_\mu^A)^2]$$

providing distribution of these variables at each point.

After rescaling  $e_\mu^A \rightarrow e_\mu^A / \mu \Delta x$  we get

$$\begin{aligned} Z[A] = & C_0 \prod_x \int de_\mu^A(x) \exp \sum_x \left[ -(e_\mu^A(x) e_\mu^A(x))^2 \right. \\ & \left. + i \frac{M^2}{\mu^2} \tilde{G}(x) \cdot \tilde{\Sigma}(x) \Delta x^2 \right], \end{aligned} \quad (5)$$

where the constant factor  $C_0 = \prod_x (\mu \Delta x)^{-16}$  is determined by the number of lattice cells,  $N = V_4 / \Delta x^4$ , in the total space volume  $V_4$ . Introducing the notation for independent averaging over tetrad at separate point,

$$\langle F \rangle = \int de_\mu^A e^{-(e_\mu^A e_\mu^A)^2} F(e_\mu^A) \quad (6)$$

the product takes the form

$$\begin{aligned} Z[A] = & C_0 \prod_x [\langle 1 \rangle + i \frac{M^2}{\mu^2} \langle \tilde{G}(x) \cdot \tilde{\Sigma}(x) \rangle \Delta x^2 \\ & - \frac{1}{2} \frac{M^4}{\mu^4} \langle (\tilde{G}(x) \cdot \tilde{\Sigma}(x))^2 \rangle \Delta x^4 + \mathcal{O}(\Delta x^4)]. \end{aligned} \quad (7)$$

An apparent  $O(16)$  symmetry of the weight integral in (6) entails simple angular averaging. Combining the index pair into a single multiple index

$\alpha = \left\{ \begin{smallmatrix} A \\ \mu \end{smallmatrix} \right\}$  we have for  $D = 16$

$$\langle e_\alpha e_\beta \rangle = \langle e^2 \rangle \delta_{\alpha\beta} \frac{1}{D} \quad (8)$$

$$\langle e_\alpha e_\beta e_\gamma e_\delta \rangle = \langle e^4 \rangle (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) \frac{1}{D(D+2)}, \quad (9)$$

$$e^2 \equiv e_\mu^A e_\mu^A, \quad e^4 \equiv (e_\mu^A e_\mu^A)^2.$$

As a consequence, the second term in (7) vanishes because of antisymmetry with respect to either space or color indices. The third term yields

$$-\frac{2}{9} \frac{M^4}{\mu^4} \langle e^4 \rangle G_{\mu\nu}^{AB}(x) G_{\mu\nu}^{AB}(x) \Delta x^4,$$

so that

$$\begin{aligned} Z[A] &= Z_0 \prod_x \left[ 1 - \frac{2}{9} \frac{M^4}{\mu^4} \frac{\langle e^4 \rangle}{\langle 1 \rangle} G^2(x) \Delta x^4 + \mathcal{O}(\Delta x^4) \right] \\ &= Z_0 \exp \left[ -\frac{2}{9} \frac{M^4}{\mu^4} \frac{\langle e^4 \rangle}{\langle 1 \rangle} \sum_x G^2(x) \Delta x^4 + \mathcal{O}(\Delta x^4) \right], \end{aligned}$$

where  $G^2 \equiv G_{\mu\nu}^{AB} G_{\mu\nu}^{AB}$  and normalization factor  $Z_0 = \prod_x [(\mu \Delta x)^{-16} \cdot \langle 1 \rangle]$ . Since the last line in the above expression is the integral sum, we finally obtain for  $\Delta x \rightarrow 0$

$$Z[A] = Z_0 \exp \left[ -\frac{2}{9} \frac{M^4}{\mu^4} \frac{\langle e^4 \rangle}{\langle 1 \rangle} \int d^4x G^2(x) \right]. \quad (10)$$

Substituting here  $\langle e^4 \rangle / \langle 1 \rangle = \int_0^\infty dr r^{19} e^{-r^4} / \int_0^\infty dr r^{15} e^{-r^4} = 4$ ,  $r^2 = e_\mu^A e_\mu^A$ , we arrive at the desired relation of the functional integral (3) to partition function of the  $SO(4)$  gauge field,

$$Z = Z_0 \int DA_\mu \exp \left[ -\frac{1}{g^2} \int d^4x G_{\mu\nu}^{AB}(A) G_{\mu\nu}^{AB}(A) \right],$$

the coupling constant being

$$\frac{1}{g^2} = \frac{8}{9} \frac{M^4}{\mu^4}. \quad (11)$$

### 3 Relation to gravity

There is a different way to work out the integral (3) starting from the Gaussian integral over gluon fields. We begin by noting that

$$\varepsilon^{\mu\nu\lambda\sigma}\varepsilon^{ABCD}\Sigma_{\lambda\sigma}^{CD} = 4 \det(e) \cdot (e^{A,\mu}e^{B,\nu} - e^{B,\mu}e^{A,\nu}) \equiv 4 \det(e) \Sigma^{AB,\mu\nu},$$

where contravariant tetrad and metric tensor  $g_{\mu\nu}$  are defined according to the relations

$$g_{\mu\nu} = e_\mu^A e_\nu^A, \quad e_\mu^A = g_{\mu\nu} e^{A,\nu}, \quad e^{A,\mu} e_\mu^B = \delta^{AB}, \quad \det g = \det(e)^2. \quad (12)$$

With these notations the second term in the exponent (3) reads

$$iM^2 \int d^4x \tilde{G}(A) \cdot \tilde{\Sigma} = 4iM^2 \int d^4x \det(e) G_{\mu\nu}^{AB}(A) \Sigma^{AB,\mu\nu}, \quad (13)$$

The expression (13) is well-known Hilbert-Palatini action (in Euclidean space), whose variation with respect  $A_\mu$  and  $e^{A,\mu}$  yields General Relativity classical equations for pure gravity [10].

It is instructive here to pursue this connection in a little bit different manner more suitable to carry out Gaussian integral. To this end we first introduce covariant derivative, which acts onto tetrad as

$$\nabla_\mu e^{A,\nu} = \omega_\mu^{CA} e^{C,\nu},$$

with the spin connection matrix  $\omega_\mu^{AB} = -\omega_\mu^{BA}$ . It automatically implies metric compatibility,  $\nabla_\lambda g_{\mu\nu} = 0$ , and allows for the obvious identity

$$\begin{aligned} & \partial_\mu [\det(e) \Sigma^{AB,\mu\nu} A_\nu^{AB}] - \partial_\nu [\det(e) \Sigma^{AB,\mu\nu} A_\mu^{AB}] \\ &= \det(e) \nabla_\mu (\Sigma^{AB,\mu\nu} A_\nu^{AB}) - \det(e) \nabla_\nu (\Sigma^{AB,\mu\nu} A_\mu^{AB}) \\ &= \det(e) [A_\nu^{AB} \nabla_\mu \Sigma^{AB,\mu\nu} - A_\mu^{AB} \nabla_\nu \Sigma^{AB,\mu\nu} + \Sigma^{AB,\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)^{AB}]. \end{aligned}$$

Furthermore, we have

$$A_\nu^{AB} \nabla_\mu \Sigma^{AB,\mu\nu} - A_\mu^{AB} \nabla_\nu \Sigma^{AB,\mu\nu} = 2 [\omega_\mu, A_\nu]^{AB} \Sigma^{AB,\mu\nu}.$$

These two identities permit the field strength tensor to be recast in the form

$$\begin{aligned} \det(e) \Sigma^{AB,\mu\nu} G_{\mu\nu}^{AB}(A) &= \partial_\mu [\det(e) \Sigma^{AB,\mu\nu} A_\nu^{AB}] - \partial_\nu [\det(e) \Sigma^{AB,\mu\nu} A_\mu^{AB}] \\ &+ \det(e) \Sigma^{AB,\mu\nu} \left( [A_\mu - \omega_\mu, A_\nu - \omega_\nu] - [\omega_\mu, \omega_\nu] \right)^{AB} \end{aligned}$$

valid for an arbitrary field  $A_\mu$ . Combining it with the same expression written for  $G_{\mu\nu}(\omega)$  we reach a net result:

$$\begin{aligned} & \det(e) \Sigma^{AB,\mu\nu} G_{\mu\nu}^{AB}(A) \\ &= \partial_\mu \left[ \det(e) \Sigma^{AB,\mu\nu} (A_\nu - \omega_\nu)^{AB} \right] - \partial_\nu \left[ \det(e) \Sigma^{AB,\mu\nu} (A_\mu - \omega_\mu)^{AB} \right] \\ &+ \det(e) \Sigma^{AB,\mu\nu} \left( [A_\mu - \omega_\mu, A_\nu - \omega_\nu] + G_{\mu\nu}(\omega) \right)^{AB}. \end{aligned} \quad (14)$$

It is worth sometimes to split second rank tensors into their self- and anti-self-dual parts with respect to flat indices,

$$T^{AB} = T^{+AB} + T^{-AB}, \quad T^{\pm AB} = \frac{1}{2} (T^{AB} \pm \varepsilon^{ABCD} T^{CD}),$$

$$[T_1^+, T_2^-] = 0, \quad T_1^{+AB} T_2^{-AB} = 0,$$

the identity  $[\tilde{T}_1, \tilde{T}_2] = [T_1, T_2]$ ,  $\tilde{T}^{AB} \equiv \varepsilon^{ABCD} T^{CD}$  being responsible for commutator vanishing in the second line. In fact, this amounts to decomposition of  $SO(4)$  algebra into two  $SU(2)$  algebras whose generators are made of plus or minus components.

Substituting  $A = A^\pm$  into equality (14) we immediately get that it holds separately for plus and minus parts of the field strength tensor,

$$\begin{aligned} & \det(e) \Sigma^{AB,\mu\nu} G_{\mu\nu}^{AB}(A^\pm) \\ &= \partial_\mu \left[ \det(e) \Sigma^{AB,\mu\nu} (A_\nu^\pm - \omega_\nu^\pm)^{AB} \right] - \partial_\nu \left[ \det(e) \Sigma^{AB,\mu\nu} (A_\mu^\pm - \omega_\mu^\pm)^{AB} \right] \\ &+ \det(e) \Sigma^{AB,\mu\nu} \left( [A_\mu^\pm - \omega_\mu^\pm, A_\nu^\pm - \omega_\nu^\pm] + G_{\mu\nu}(\omega^\pm) \right)^{AB} \end{aligned} \quad (15)$$

provided we take into account that  $G_{\mu\nu}^{AB}(A^\pm) = G_{\mu\nu}^{\pm AB}(A)$ .

By virtue of the identity (14) the Gaussian integral over  $A$  in (3) can be trivially done by replacement  $(A - \omega) \rightarrow \bar{A}$ . Moreover since the quadratic in fields  $\bar{A}_\mu$  part of the action (13),

$$\begin{aligned} i\bar{A} \cdot \mathcal{K} \cdot \bar{A} &= 4iM^2 \int d^4x \det(e) \Sigma^{AB,\mu\nu} [\bar{A}_\mu, \bar{A}_\nu]^{AB} \\ &= 4iM^2 \int d^4x \det(e) \Sigma^{AB,\mu\nu} ([\bar{A}_\mu^+, \bar{A}_\nu^+] + [\bar{A}_\mu^-, \bar{A}_\nu^-])^{AB} \end{aligned} \quad (16)$$

does not contain derivatives, the functional determinant decays into infinite product of usual determinants at each space point,

$$\text{Det}(\mathcal{K}/\pi)^{-\frac{1}{2}} = \prod_x c \det[e(x)]^{-6} [M^2 \Delta x^4]^{-12} \quad (17)$$

( $c = 2^{-28} \pi^{12}$ ), producing an additional local factor for the functional measure  $De_\mu^A$ . Thus the integral (3) takes the form

$$Z = \int De_\mu^A \rho(e_\mu^A) \exp \int d^4x \left[ -\mu^4 (e_\mu^A e_\mu^A)^2 + 4iM^2 G_{\mu\nu}^{AB} \Sigma^{AB,\mu\nu} \right]. \quad (18)$$

Recalling that

$$4iM^2 \int d^4x G_{\mu\nu}^{AB} \Sigma^{AB,\mu\nu} = 8iM^2 \int d^4x R \det e$$

looks like the conventional Einstein-Hilbert action, this form bears a close resemblance to gravity partition function. A straightforward identification with it requires, however, a physical interpretation of the first term, which explicitly spoils general covariance of the action appearing in the integral (18). Further, the space volume in the Einstein-Hilbert action should be positive,  $\sqrt{g} = |\det(e)|$ , which has to impose certain restrictions on the integrals over tetrad. Another possible question is a proper choice of the local factor  $\rho(e_\mu^A)$ . These issues will be discussed in the next two sections.

## 4 Gravity measure

For proper description of quantum gravity the functional (3) needs somewhat modification, namely, the functional measure  $De_\mu^A$  has to be changed by extra local factor.

There are several alternative ways to define its form.

Similarly to gauge fields theories, where the measure  $DA_\mu$  is invariant under gauge transformation, the relevant gravity measure  $Dg$  is supposed to be invariant under general coordinate transformation. The invariance is achieved by inclusion of a local factor [11],

$$Dg = \prod_x \prod_{\mu \leq \nu} g^{\frac{5}{2}} dg^{\mu\nu} = \prod_x \prod_{\mu \leq \nu} g^{-\frac{5}{2}} dg_{\mu\nu}, \quad (19)$$

$$g = \det(g_{\mu\nu}).$$



Another form has been obtained in the hamiltonian formalism, which ensures unitarity of  $S$ -matrix for gravity [12],

$$Dg = \prod_x \prod_{\mu \leq \nu} g^{-\frac{3}{2}} g^{00} dg_{\mu\nu} \quad (20)$$

The hamiltonian formalism applied to the Palatini-Holst action leads to the measure [13], which in our notations reads

$$DA De = \prod_x dA_\mu^{AB} de_\mu^A \mathcal{V}^3 V_s, \quad (21)$$

where  $\mathcal{V} = \sqrt{g}$  is a spacetime volume element while  $V_s$  is a spatial volume element.

Actually the local factor in the measures affects only high orders of the perturbation theory acting as counterterm that cancels the divergent pieces of the loops proportional to  $\delta^{(4)}(0)$  [12]. The measures (20, 21) are not invariant under coordinate transformations. The reason for this lies in the procedure of quantization, namely, in a spacetime lattice that is implied behind path integral. It provides an ultraviolet regularization but violates invariance to the coordinate transformation [14].

Despite the modifications the steps (4), (5), (7) hold unchanged provided the averaging is redefined according to the measures (19), (20), (21). Particular form of the local factors results only in different coefficients in the equation for the coupling constant (11). To show it we begin with the measure

$$DA De = \prod_x dA_\mu^{AB} de_\mu^A \det(e)^K \quad (22)$$

The local factor in (22) requires to redefine the averaging over tetrad as

$$\langle F \rangle_K = \int de_\mu^A \det(e)^K e^{-(e_\mu^A e_\mu^A)^2} F(e_\mu^A). \quad (23)$$

The additional factor  $\det(e)^K$  breaks  $O(16)$  symmetry and invalidates the identities (8), (9). But it allows for independent  $O(4)$  rotations of color and space tetrad indices, that results into the set of relations instead of (8), (9):

$$\langle e_\mu^A e_\nu^B \rangle_K = \delta^{AB} \delta_{\mu\nu} \frac{1}{16} \langle g_{\alpha\alpha} \rangle_K, \quad (24)$$

$$\langle e_\mu^A e_\nu^B e_\lambda^C e_\sigma^D \rangle_K = \delta^{AB} \delta^{CD} s_{\mu\nu\lambda\sigma} + \delta^{AC} \delta^{BD} s_{\mu\lambda\nu\sigma} + \delta^{AD} \delta^{BC} s_{\mu\sigma\nu\lambda}, \quad (25)$$

with

$$s_{\mu\nu\lambda\sigma} = \frac{1}{576} [(3g_1^2 - 2g_2)\delta_{\lambda\sigma}\delta_{\mu\nu} - (g_1^2 - 2g_2)(\delta_{\lambda\mu}\delta_{\nu\sigma} + \delta_{\lambda\nu}\delta_{\mu\sigma})] \quad (26)$$

and

$$g_1^2 = \langle g_{\alpha\alpha}g_{\beta\beta} \rangle_K, \quad g_2 = \langle g_{\alpha\beta}g_{\beta\alpha} \rangle_K. \quad (27)$$

The first identity (24) removes the  $\Delta x^2$  term in the expansion (7), while the second one (25) returns the value of the term proportional to  $\Delta x^4$ , which yields in the continuous limit

$$Z[A] = Z_0 \exp \left[ -\frac{4}{9} \frac{M^4}{\mu^4} \frac{g_1^2 - g_2}{\langle 1 \rangle_K} \int d^4x G^2(x) \right], \quad (28)$$

$$Z_0 = \prod_x [(\mu\Delta x)^{-16-4K} \langle 1 \rangle_K].$$

Using  $g_1^2 - g_2$  value calculated in the Appendix we obtain the coupling constant

$$\frac{1}{g^2} = \frac{2}{3} \frac{M^4}{\mu^4} \frac{(K+3)(K+4)}{2K+9}. \quad (29)$$

and arrive at the relation

$$\begin{aligned} & \int De \exp \int d^4x \left[ -\mu^4 (e_\mu^A e_\mu^A)^2 + 8iM^2 R \det(e) \right] \\ &= Z_{gA} \int DA \exp \left[ -\frac{1}{g^2} \int d^4x G_{\mu\nu}^{AB}(A) G_{\mu\nu}^{AB}(A) \right]. \end{aligned} \quad (30)$$

The functional measure is understood as

$$De = \prod_x \rho(e_\mu^A) de_\mu^A,$$

with the local factor encountering the determinant (17),  $\rho(e_\mu^A) = \det(e)^{K-6}$ . The "transition" coefficient between gravity and gauge partition function is given by the product running over space points,

$$Z_{gA} = \prod_x c_e M^{-24} \left[ (\mu^4 \Delta x^4)^{-4-K} \langle 1 \rangle_K (M^4 \Delta x^4)^{12} \right], \quad (31)$$

$$c_e = 2^{28} \pi^{-12}.$$

Now we have to correct this result for the condition  $\det(e) \geq 0$  assumed in the gravity action. One way to incorporate it is to integrate over configurations for which the inequality holds, that is to choose instead of the measure (22) the modified one,

$$DA De^+ = \prod_x dA_\mu^{AB} de_\mu^A \det(e)^K \theta(\det(e)). \quad (32)$$

Indeed, the functional, coming about upon the integration over gauge field  $A_\mu$ , depends only on the metric tensor, the curvature, that is expressed through the metric tensor again, and  $\det(e)$ . The tensor  $g_{\mu\nu} = e_\mu^A e_\nu^A$  remains unchanged under reflection of sign in any row of the matrix  $e_\mu^A$ , that negates  $\det(e)$ . Therefore for an arbitrary function of these variables one may write

$$\int de_\mu^A \theta(\det(e)) f(\det(e), g_{\mu\nu}) = \frac{1}{2} \int de_\mu^A f(\sqrt{g}, g_{\mu\nu}). \quad (33)$$

It allows to pass from tetrad to the integral over  $g_{\mu\nu}$  with the measure

$$Dg = \prod_x \prod_{\mu \leq \nu} g^N dg_{\mu\nu}. \quad (34)$$

The power  $N$  is separated here into two parts. The first part is due to the determinant (17), the second one arises as a local Jacobian, since for any function  $f$

$$\int \prod_{\mu, A} de_\mu^A f(e_\mu^A e_\nu^A) = \int \prod_{\mu \leq \nu} dg_{\mu\nu} g^{-\frac{1}{2}} f(g_{\mu\nu})$$

(see the Appendix). Putting it together with  $\det(e)^K$  we recover the measure (34) for  $K = 2N + 7$ .

The measure (32) admits  $SO(4) \times SO(4)$  independent rotations of color and space indices rather than  $O(4) \times O(4)$  before. This restriction allows for one more term in the equation (25),

$$\begin{aligned} \langle e_\mu^A e_\nu^B e_\lambda^C e_\sigma^D \rangle_K &= \delta^{AB} \delta^{CD} s_{\mu\nu\lambda\sigma} + \delta^{AC} \delta^{BD} s_{\mu\lambda\nu\sigma} + \delta^{AD} \delta^{BC} s_{\mu\sigma\nu\lambda} \\ &+ \frac{1}{24} \varepsilon^{ABCD} \varepsilon_{\mu\nu\lambda\sigma} \langle \det(e) \rangle_K, \end{aligned} \quad (35)$$

and the constraint  $\det(e) \geq 0$  amounts to an extra contribution to the gauge action,

$$\begin{aligned} Z[A] &= Z_0 \exp \int d^4x \left[ -\frac{1}{g^2} G_{\mu\nu}^{AB}(A) G_{\mu\nu}^{AB}(A) \right. \\ &\quad \left. + \tilde{c} \varepsilon^{ABCD} \varepsilon_{\mu\nu\lambda\sigma} G_{\mu\nu}^{AB}(A) G_{\lambda\sigma}^{CD}(A) \right], \end{aligned}$$

This term appears to be a total derivative with the constant  $\tilde{c}$  in front given by the averaged 'volume element'

$$\tilde{c} = \frac{4}{3} \frac{\langle |\det(e)| \rangle_K}{\langle 1 \rangle_K} \frac{M^4}{\mu^4},$$

which for the particular form (32) evaluates to  $\tilde{c} = \frac{1}{3\sqrt{\pi}} \frac{2K+1}{2K+9}$ .

Consider now the measures (20) and (21). They share the common lack of explicit covariance because of their hamiltonian nature that distinguishes 'time' and 'space'. It breaks the  $O(4)$  symmetry of the local averaging with respect to the space indices though the  $O(4)$  or  $SO(4)$  color symmetry remains intact, so that the relations (24) and (25) or (35) are still applicable while the formula (26) should be changed. Introducing 'time' directed unit vector  $n_\mu$  one can write instead of it

$$s_{\mu\nu\lambda\sigma} = c_1 \delta_{\mu\nu} \delta_{\lambda\sigma} + c_2 (\delta_{\lambda\mu} \delta_{\nu\sigma} + \delta_{\lambda\nu} \delta_{\mu\sigma}) + c_3 n_\mu n_\nu n_\lambda n_\sigma, \quad (36)$$

the coefficients being again expressed through averaged values (27) and a new structure  $g_{nn} = \langle n_\alpha g_{\alpha\beta} n_\beta \rangle_K$ ,

$$\begin{aligned} c_1 &= \frac{1}{1512} (8g_1^2 - 5g_2 - 3g_{nn}), & c_2 &= -\frac{1}{3024} (5g_1^2 - 11g_2 - 6g_{nn}), \\ c_3 &= -\frac{1}{504} (g_1^2 + 2g_2 - 24g_{nn}), \end{aligned}$$

where the averaging is carried out with particular local factors determining the measure. The output gluon partition function has the same form (28), in which the details of gravity measure are encoded only in the parameters  $g_1^2$ ,  $g_2$ ,  $\langle 1 \rangle$  and  $\langle |\det(e)| \rangle$ . The last one defines the total derivative term appearing in the gluon action, when the constraint  $\det(e) > 0$  is imposed. The term  $g_{nn}$  does not contribute.

Thus the variation of the functional measure affects the final relation between gravity and gauge field,

$$\begin{aligned} & \int Dg \exp \int d^4x \left[ -\mu^4 g_{\mu\mu}^2 + 8iM^2 R\sqrt{g} \right] \\ &= Z_{gA} \int DA \exp \left[ -\frac{1}{g^2} \int d^4x G_{\mu\nu}^{AB}(A) G_{\mu\nu}^{AB}(A) + \tilde{c} \int d^4x G_{\mu\nu}^{AB}(A) \tilde{G}_{\mu\nu}^{AB}(A) \right], \end{aligned} \quad (37)$$

only through coupling  $g$ , constant  $\tilde{c}$  and normalization product (31) (with  $c_e = 2^{29}\pi^{-12}$  because of 1/2 in (33)). The average  $\langle 1 \rangle_K$  is calculated for

a given local factor  $\rho(g)$  in the gravity measure. It is supposed to be homogenous function for the metrics rescaling,  $g_{\mu\nu} \rightarrow \alpha g_{\mu\nu}$ ,  $\rho(g) \rightarrow \alpha^{4N} \rho(g)$ ,  $K = 2N + 7$ .

## 5 Duality

The second term in the l.h.s. (37) is the conventional Einstein-Hilbert action that enjoys general coordinate invariance in contrast to the first term that explicitly violates it. To clarify a meaning behind it we consider partition function for the pure Euclidean gravity without extra terms,

$$Z_g = \int Dg \exp \int d^4x \left[ 8iM^2 R \sqrt{g} \right].$$

This functional is to be supplemented with appropriate gauge conditions fixing in gravity case the coordinate system. The four coordinates should be fixed with four constraints imposed on the metric tensor. Let us choose as variables subject to the constraints the diagonal components of  $g_{\mu\mu}$ ,

$$g_{\mu\mu}(x) = \alpha_\mu(x). \quad (38)$$

According to the standard proceeding it amounts to dealing with gauge-fixed integral

$$Z_g^{gf} = \int Dg \Delta_{FP}[g] \prod_\mu \delta[g_{\mu\mu} - \alpha_\mu] \exp \int d^4x \left[ 8iM^2 R \sqrt{g} \right]$$

with Faddeev-Popov determinant  $\Delta_{FP}[g]$ . Two integrals,  $Z_g$  and  $Z_g^{gf}$  differ only in the constant normalization proportional to the volume of gauge group that is the group of coordinate diffeomorphisms, in our case. The first order variation of the gauge conditions under the infinitesimal action of this group,

$$\delta x_\mu = \epsilon_\mu(x), \quad \delta g_{\mu\mu}(x) = -2\nabla_\mu \epsilon_\mu(x)$$

yields Faddeev-Popov determinant,

$$\Delta_{FP}[g] = \prod_\mu \text{Det}(\nabla_\mu).$$

Expressing covariant derivative through Christoffel symbols,  $(\nabla_\mu)_\beta^\alpha = \partial_\mu \delta_\beta^\alpha + (\Gamma_\mu)_\beta^\alpha$ , we rewrite the determinant as

$$\text{Det} \partial_\mu \cdot \text{Det} [1 + \theta \cdot \Gamma_\mu],$$

where  $\partial_\mu \theta(x_\mu - y_\mu) = \delta(x_\mu - y_\mu)$ , or

$$\text{Det}(\nabla_\mu) = \text{Det} \partial_\mu \cdot \exp\left\{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr} [\theta \cdot \Gamma_\mu]^n\right\}.$$

Apart the index summation the trace implies the loop integrals over coordinates, which all are zero because of the arguments ordering in  $\theta(x - y)$  function. Thus  $\text{Det}(\nabla_\mu) = \text{Det} \partial_\mu$  is a constant not depending on the metric tensor  $g_{\mu\nu}$ . This property is similar to ghost decoupling well known for axial or planar gauges in QCD [15]. Furthermore, the functional  $Z_g^{gf}$  does not depend on the functions  $\alpha_\mu$  since its variation only "moves the point along the same gauge group orbit". It allows to average  $Z_g^{gf}$  over  $\alpha_\mu$  with arbitrary weight factor that results only in an overall constant in front,

$$Z_g^{gf} = N \int D\alpha_\mu \Phi[\alpha] \int Dg \prod_\mu \delta[g_{\mu\mu} - \alpha_\mu] \exp \int d^4x \left[ 8iM^2 R \sqrt{g} \right] \quad (39)$$

Choosing

$$\Phi[\alpha] = \exp[-\mu^4 \int d^4x (\sum_\nu \alpha_\nu)^2]$$

we arrive at the functional (37). One can conclude therefore that there is a duality between the  $SO(4)$  gauge theory and quantum gravity taken in the particular gauge (38).

## 6 Further implications

The weight function in the integral (3) may be viewed, by itself, as a part of local factor in the measure,

$$Z = \int DA_\mu D e_\mu^A \rho(e_\mu^A) \exp i \int d^4x M^2 \tilde{G}(A) \cdot \tilde{\Sigma}, \quad (40)$$

$$\rho(e_\mu^A) = \exp\{-(\mu\Delta x)^4 (e_\mu^A e_\mu^A)^2\}.$$

This form suggests a natural extension to other weight functions that admit continuous limit, for example,

$$\rho(e_\mu^A) = \exp\{-(\mu\Delta x)^4 (e_\mu^A e_\mu^A)^2\} + c \exp\{-\alpha(\mu\Delta x)^4 (e_\mu^A e_\mu^A)^2\}.$$

The formulae (10), (28) still hold in this case if the average values (6), (23) appearing in them are modified in the same manner, that is  $e^{-(e_\mu^A e_\mu^A)^2} \rightarrow$

$e^{-(e_\mu^A e_\mu^A)^2} + ce^{-\alpha(e_\mu^A e_\mu^A)^2}$ . Similarly, taking the functional  $\Phi[\alpha]$  in (39) as infinite product of local terms,

$$\Phi[\alpha] = \prod_{x,\mu} \rho(e_\mu^A),$$

we again reproduce gauge/gravity duality interpreting noncovariant weight factor in the measure as an ingredient of gauge fixing procedure for the gravitational field.

Another interesting perspective comes about if we replace the seed functional integral  $Z[A]$  (3) by the expression,

$$Z_\varepsilon[A] = \int DA_\mu D e_\mu^A \det(e)^K e^{S_\varepsilon} \quad (41)$$

with a new action,

$$S_\varepsilon = \int d^4x \left[ -\mu^4 (e_\mu^A e_\mu^A)^2 + iM^2 \det(e) \varepsilon^{ABCD} \Sigma^{AB,\mu\nu} G_{\mu\nu}^{CD} \right],$$

and arbitrary power  $K$  in the functional measure. The action of the form

$$S_H = \frac{1}{4} \int d^4x \det(e) \Sigma^{AB,\mu\nu} \left[ G_{\mu\nu}^{AB} - \frac{1}{2\gamma} \varepsilon^{ABCD} G_{\mu\nu}^{CD} \right]$$

is generalized Hilbert-Palatini action proposed by Holst [16]. It gives rise to the same equation of motion for classical gravity regardless the value of Immirzi parameter  $\gamma$  [17] (though it may affect quantum theory [18]). It is this second, Immirzi related term, in the Holst action that is only left in the action  $S_\varepsilon$  in (41).

Proceeding as before and integrating over tetrad with the help of equalities

$$\det(e) \varepsilon^{ABCD} \Sigma^{CD,\mu\nu} = \varepsilon^{\mu\nu\lambda\sigma} \Sigma_{\lambda\sigma}^{AB}$$

and (24), (25) we draw the connection of integral (41) to the partition function of  $SO(4)$  gauge field similar to (28),

$$\begin{aligned} Z_\varepsilon[A] &= \prod_x \left[ (\mu \Delta x)^{-16-4K} \langle 1 \rangle_K \right] \\ &\times \int DA \exp \left\{ -\frac{1}{9} \frac{M^4}{\mu^4} \frac{g_1^2 - g_2}{\langle 1 \rangle_K} \int d^4x G^2 \right\}, \end{aligned} \quad (42)$$

which amounts to the coupling constant value

$$\frac{1}{g^2} = \frac{1}{6} \frac{(K+3)(K+4)}{2K+9} \frac{M^4}{\mu^4}. \quad (43)$$

On the other hand, presenting dual tensor as

$$\det(e) \varepsilon^{ABCD} \Sigma^{AB,\mu\nu} G_{\mu\nu}^{CD} = \det(e) [G_{\mu\nu}^{+AB} - \bar{G}_{\mu\nu}^{AB}] \Sigma^{AB,\mu\nu},$$

and recalling the identities (15), we bring the action that appears in (41) to the form (omitting total derivatives)

$$\begin{aligned} S_\varepsilon = & \int d^4x \left[ -\mu^4 (e_\mu^A e_\mu^A)^2 + iM^2 \det(e) \Sigma^{AB,\mu\nu} \left( [A_\mu^+ - \omega_\mu^+, A_\nu^+ - \omega_\nu^+] \right. \right. \\ & \left. \left. - [A_\mu^- - \omega_\mu^-, A_\nu^- - \omega_\nu^-] \right)^{AB} + iM^2 \det(e) \Sigma^{AB,\mu\nu} [G_{\mu\nu}^+(\omega) - \bar{G}_{\mu\nu}(\omega)]^{AB} \right]. \end{aligned}$$

The last term is identically zero here, as immediately follows from the Bianchi identity for curvature tensor,

$$\begin{aligned} \det(e) \Sigma^{AB,\mu\nu} \varepsilon^{ABCD} G_{\mu\nu}^{CD}(\omega) &= \frac{1}{2} \det(e) \Sigma^{AB,\mu\nu} R_{\lambda\sigma\mu\nu} \Sigma^{CD,\lambda\sigma} \\ &= \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \Sigma_{\alpha\beta}^{CD} R_{\lambda\sigma\mu\nu} \Sigma^{CD,\lambda\sigma} = 2\varepsilon^{\mu\nu\lambda\sigma} R_{\lambda\sigma\mu\nu} = 0. \end{aligned}$$

As a consequence the replacement  $A_\mu - \omega_\mu \rightarrow A_\mu$  completely removes all derivatives from  $S_\varepsilon$ , so that the whole integral (41) turns into product of independent integrals uncorrelated at each space point both for tetrad and gauge fields,

$$\begin{aligned} Z_\varepsilon[A] = & \int DA_\mu D\omega_\mu^A \det(e)^K \exp \int d^4x \left[ -\mu^4 (e_\mu^A e_\mu^A)^2 \right. \\ & \left. + iM^2 \det(e) \Sigma^{AB,\mu\nu} \left( [A_\mu^+, A_\nu^+] - [A_\mu^-, A_\nu^-] \right)^{AB} \right]. \end{aligned} \quad (44)$$

The form (44) entails, in particular, a closed expression of the same type for the gluon partition function  $Z[A]$ . Indeed, comparing (44) with (41) and (42), we present  $Z[A]$  through uncorrelated product as

$$Z[A] = \prod_x \int dA \exp \left\{ -\frac{1}{g^2} [A_\mu, A_\nu]^{AB} [A_\mu, A_\nu]^{AB} (\Delta x)^4 \right\}. \quad (45)$$

This simple result is plagued by non-decreasing of the integrand along  $A_\mu = A_\nu$  directions, which makes it divergent at each space point  $x$ . However it is



finite if the integrals in (44) are taken in opposite order. Integrating out at first the gauge fields yields (after rescaling  $e_\mu^A \rightarrow e_\mu^A/(\mu\Delta x)$ )

$$Z_\varepsilon = \prod_x (\mu\Delta x)^{-16-4K} \left[ \int de_\mu^A \det(e)^K e^{-(e_\mu^A e_\mu^A)^2} \left( \det \frac{\tilde{\mathcal{K}}}{\pi} \right)^{-\frac{1}{2}} \right],$$

the matrix  $\tilde{\mathcal{K}}$  defining quadratic form in the Gaussian integral,

$$iA \cdot \tilde{\mathcal{K}} \cdot A = iM^2 \int d^4x \det(e) \Sigma^{AB, \mu\nu} \left( [A_\mu^+, A_\nu^+] - [A_\mu^-, A_\nu^-] \right)^{AB}.$$

This form differs from that appearing in (16) only in the relative sign between the positive and negative blocks and the overall coefficient in front. Since both matrices,  $\mathcal{K}$  and  $\tilde{\mathcal{K}}$ , are block diagonal, their determinants are proportional, and

$$Z_\varepsilon[A] = \prod_x \left[ (\mu\Delta x)^{-16-4K} \left( \frac{M}{\mu} \Delta x \right)^{-24} \langle 1 \rangle_{K-6} \frac{\pi^{12}}{16} \right].$$

Comparing again this expression with (42) we arrive at the "finite" result for the gluon partition function,

$$Z[A] = \prod_x \left[ \frac{\pi^{12}}{16} \left( \frac{M}{\mu} \Delta x \right)^{-24} \frac{\langle 1 \rangle_{K-6}}{\langle 1 \rangle_K} \right]. \quad (46)$$

Substituting here explicit expressions for  $\langle 1 \rangle_K$  from the Appendix and the coupling constant (43) we get

$$Z[A] = \prod_x \left[ g^{12} (\Delta x)^{-24} c_K \right] \quad (47)$$

with  $c_K = 4\pi^{15} \left[ \frac{(K+3)(K+4)}{3(2K+9)} \right]^6 \frac{(2K+7)(2K+5)(4K^2-9)(4K^2-1)}{(K-1)(K-3)(K-5)}$ .

Thus the divergency in (45) may be regarded as a consequence of the continuous limit that has not been assumed in deriving (46). Obviously, the divergency has an ultraviolet origin, since it appears when  $\Delta x \rightarrow 0$ . The equation (46) is then natural to treat as being obtained with a kind of lattice regularization characterized, besides the fixed  $\Delta x$ , by the parameter  $K$ .

## 7 Conclusion

The above treatment can be summarized in the three main statements:

1. There is a simple connection between partition functions of gravity with an extra noncovariant term added to Einstein-Hilbert action and  $SO(4)$  gauge theory (30), (37).
2. The noncovariant part of the gravity action in (30), (37) is natural to interpret as the gauge-fixing term for a particular gauge (38) imposed on the metric tensor.
3. The partition function of  $SO(4)$  gauge theory can be brought to the form, in which the action does not contain fields derivatives, and the functional integral reduces to the product of independent finite dimensional integrals at each space points (45), (46).

The basic method to find gauge/gravity connection relies on the equation (14). By shifting  $A_\mu \rightarrow A_\mu - \omega_\mu$  it removes the derivatives either from gauge field or tetrad. The functional integral without derivatives looks like averaging over ensemble of uncorrelated random variables. According to the "large numbers law" the result is weakly sensitive to the distribution of single variables, being completely determined with a few parameters like mean value and dispersion accumulating the details. That is why any particular functional measure chosen for quantum gravity leads to the same standard action for the gauge field changing the coupling constant(s) only. On the other hand recasting derivatives onto tetrad yields Einstein-Hilbert gravity action (with fixed gauge) while the gauge field turns into uncorrelated ensemble and, having been integrated out, produces the additional local factor for the gravity measure (17).

The equation (15) develops this even further completely removing derivatives both from the tetrad and the field  $A_\mu$  without giving rise to a "gravity". It makes the gluon partition function to be entirely uncorrelated like that for the lattice with no interaction between neighbor space points.

There are two comments in order here. First, the replacement  $A_\mu \rightarrow A_\mu - \omega_\mu$  removes derivatives only in the functional integral for the gluon partition function but does not work for more complex objects such as, say, the Green functions. The correlations do not disappear for the action with external source.

Second, the divergency of the continuous partition function (45) calls for ultraviolet regularization provided with the finite spacing  $\Delta x$  and local factor in the measure (41). The regularized result (46), that includes apart from

these two parameters the bare charge  $g$ , could be significantly influenced by subsequent renormalization. However even if possible corrections are considerable they originate from short distances and therefore should be in the perturbative region of the gauge theory.

## 8 Appendix

Here we briefly comment the computation of the integrals of the form

$$I_m = \int de_\mu^A \det(e)^m e^{-(e_\mu^A e_\mu^A)^2} \quad (48)$$

encountered in the above treatment. It is convenient to start from a bit more general integral

$$I = \int de_i^A F(e_i^A e_k^A),$$

when the variable  $e_i^A$  carries space and color indices in the intervals  $i = 1, \dots, D$ ,  $A = 1, \dots, N_A$  respectively, and  $F$  is arbitrary function. Inserting auxiliary integral over symmetric matrix  $g_{ik}$ ,

$$I = \int de_i^A F(e_i^A e_k^A) = \int \prod_{i \leq k} dg_{ik} \int de_i^A \delta(e_i^A e_k^A - g_{ik}) F(g_{ik}),$$

where

$$\delta(X) \equiv \prod_{i \leq k} \delta(X_{ik})$$

for any symmetric matrix  $X$ , and using the relation

$$\delta(C \cdot X \cdot C) = \frac{1}{\det C^{D+1}} \delta(X)$$

valid for any symmetric matrix  $C$ , we transform integral to the form

$$I = \int \prod_{i \leq k} dg_{ik} F(g_{ik}) \frac{1}{\det g^{\frac{D+1}{2}}} \int de_i^A \delta(g^{-\frac{1}{2}} e^A e^A g^{-\frac{1}{2}} - 1),$$

in which  $(e^A e^A)_{ik} \equiv e_i^A e_k^A$  and positivity of  $g$  assures  $g^{\frac{1}{2}}$  existence. Changing the variables  $e_i^A = g_{ik}^{\frac{1}{2}} \bar{e}_k^A$ , we finally obtain

$$I = J_{N_A} \int \prod_{i \leq k} dg_{ik} F(g_{ik}) \det g^{\frac{N_A - D - 1}{2}} \quad (49)$$

with the factor

$$J_{N_A} = \int d\bar{e}_i^A \delta(\bar{e}^A \bar{e}^A - 1) \quad (50)$$

independent on  $g$  and  $F$ . The formula (49) for  $N_A = D$  reproduces Jacobian  $\det g^{-\frac{1}{2}}$  used in the derivation of the measure (22).

To find  $J_N$  we note firstly that

$$J_{N+1} = \int de_k^{N+1} \int de_i^A \delta(e_i^A e_k^A - g_{ik})$$

with  $g_{ik} = \delta_{ik} - e_i^{N+1} e_k^{N+1}$ , and secondly that  $|e_i^{N+1}| \leq 1$  in this integral. Since  $\det g = 1 - e_i^{N+1} e_i^{N+1}$  we have the recursion equation

$$\begin{aligned} J_{N+1} &= J_N \int de_k^{N+1} (1 - e_i^{N+1} e_i^{N+1})^{\frac{N-D-1}{2}} \\ &= J_N \Omega_D \int_0^1 dr r^{D-1} (1 - r^2)^{\frac{N-D-1}{2}} = J_N \Omega_D B\left(\frac{1}{2}, \frac{N-D+1}{2}\right), \end{aligned}$$

whose solution reads

$$J_{D+N} = \left[\frac{1}{2} \Omega_D\right]^N \frac{\Gamma^{N+1}\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{N}{2}\right)} J_D.$$

Turning back to the integral (48) we see that it can be calculated through the following chain of equalities resembling a kind of replica method

$$\begin{aligned} I_m &= J_D \int \prod_{i \leq k} dg_{ik} e^{-g_{ii}^2} \det g^{\frac{m-1}{2}} = \frac{J_D}{J_{D+m}} \int \prod_{A=1}^{D+m} de_i^A e^{-(e_i^A e_i^A)^2} \\ &= \frac{J_D}{J_{D+m}} \Omega_{D+m} \int_0^\infty dr r^{D+m-1} e^{-r^4} = \frac{1}{4} \frac{J_D}{J_{D+m}} \Omega_{D+m} \Gamma\left(\frac{D+m}{4}\right), \end{aligned}$$

More general integrals of the form  $\int de_\mu^A (Tr g^k)^n \det(e)^m e^{-(e_\mu^A e_\mu^A)^2}$  are calculated with the same trick and the help of relations of the type (8),(9) with  $D \rightarrow D+m$ , the factors  $\frac{J_4}{J_{N+2m}} \Omega_{D+m}$  canceling in the ratio  $\langle (Tr g^k)^n \rangle / \langle 1 \rangle$ .

Taking  $D = 4$  and  $m = K$  we get

$$\begin{aligned} \langle 1 \rangle_K &= \frac{J_4}{J_{K+4}} \Omega_{4(4+K)} \frac{1}{4} \Gamma(K+4), \\ \langle e^4 \rangle_K &= \frac{J_4}{J_{K+4}} \Omega_{4(4+K)} \frac{1}{4} \Gamma(K+5), \end{aligned}$$

$e^4 \equiv (e_i^A e_i^A)^2$ , whereas

$$g_1^2 - g_2 = \frac{3}{2} \frac{K+3}{2K+9} \langle e^4 \rangle_K.$$

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